

A Maximum-Principle-Satisfying High-order Finite Volume Compact WENO Scheme for Scalar Conservation Laws

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Abstract

In this paper, a maximum-principle-satisfying finite volume compact scheme is proposed for solving scalar hyperbolic conservation laws. The scheme combines WENO schemes (Weighted Essentially Non-Oscillatory) with a class of compact schemes under a finite volume framework, in which the nonlinear WENO weights are coupled with lower order compact stencils. The maximum-principle-satisfying polynomial rescaling limiter in [41] is adopted to construct the present schemes at each stage of an explicit Runge-Kutta method, without destroying high order accuracy and conservativity. Numerical examples for one and two dimensional problems including incompressible flows are presented to assess the good performance, maximum principle preserving, essentially non-oscillatory and highly accurate resolution of the proposed method.

Keywords: compact scheme; finite volume WENO; maximum-principle-satisfying; scalar hyperbolic conservation laws; incompressible flow

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1 Introduction

In this paper we consider the following scalar hyperbolic conservation equation

$$\frac{\partial u(x, t)}{\partial t} + \nabla \cdot f(u(x, t)) = 0, \quad x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d, \quad (1.1)$$

with the initial condition $u(0, x) = u_0(x)$. A main property of (1.1) is that the solution $u(x, t)$ might develop discontinuities in finite time even when the initial data is smooth, due to its different propagation speeds. The weak solutions of (1.1) are not unique, hence a physically relevant entropy solution should be considered. An important property of the entropy solution for (1.1) is that it satisfies a strict maximum-principle [6], namely

$$u_m \leq u(x, t) \leq u_M, \quad \text{if } u_m \leq u(x, 0) \leq u_M, \quad (1.2)$$

where $u_m = \min_x u(x, 0)$ and $u_M = \max_x u(x, 0)$. The total variation diminishing (TVD) schemes [16] satisfy the strict maximum-principle, but they degenerate to first order accuracy at smooth extrema [28]. Recently Zhang et. al. proposed uniformly high order accurate schemes satisfying a strict maximum-principle based on the finite volume weighted essentially non-oscillatory (WENO) and finite element discontinuous Galerkin (DG) frameworks for scalar conservation laws [41, 43]. These high order schemes achieve the strict maximum principle by applying a polynomial rescaling limiter at each stage of an explicit Runge-Kutta (RK) method or at each step of a multistep method. The technique was later generalized to positivity preserving high order DG and finite volume WENO schemes for compressible Euler equations [42]. Another class of high order parametrized maximum-principle-preserving (MPP) flux limiters was proposed by Xu et. al. [39, 25] under a finite volume framework, which limits a high order numerical flux towards a first order monotone flux. Later in [38] Xiong et. al. generalized the parametrized high order MPP flux limiters for finite difference RK-WENO schemes with applications in incompressible flows. They only applied the limiters at the final stage of an explicit RK method which could save much computational cost. For these finite difference and finite volume WENO schemes solving scalar hyperbolic conservation laws, they are high order accurate in smooth regions and essentially non-oscillatory for shock capturing. However, these methods based on non-compact WENO schemes often suffer from excessive numerical dissipation, poor spectral resolution and increasingly wide stencils with increasing order of accuracy.

A class of compact finite difference schemes was proposed in [22], which have significant higher spectral resolutions with narrower stencils. The compact scheme has been applied to

incompressible flows in [9, 37] and compressible flows in [23, 10]. However this classical linear compact finite difference scheme often yields oscillatory solutions across discontinuities. To address this difficulty, several hybrid schemes are proposed to couple the ENO and WENO schemes for simulating shock-turbulence interaction problems, e.g., a hybrid compact-ENO scheme by Adams et. al. [1] and a hybrid compact WENO scheme by Pirozzoli [33]. Ren et. al. proposed a characteristic-wise hybrid compact WENO scheme [35] as a weighted average of the conservative compact scheme [33] and the WENO scheme [18]. These hybrid schemes used smooth indicators to transit from the compact scheme to the WENO scheme around the discontinuities.

As an alternative to the hybrid schemes, a nonlinear compact scheme was proposed by Cockburn and Shu [5] based on TVD and TVB limiters. Deng et. al. then developed fourth order and fifth order weighted compact nonlinear schemes from cell-centered compact schemes and compact interpolations of conservative variables at cell edges [7, 8]. The idea was generalized to weighted compact nonlinear schemes with increasingly high order accuracies in [40] by directly interpolating the flux. However, these schemes are not truly compact schemes and the spectral resolution would be reduced. A new class of linear central compact scheme with spectral-like resolution was recently proposed in [27] based on the cell-centered compact scheme of Lele [22]. Instead of interpolating the values on cell centers, they directly evolve the values both on the grid nodes and the cell centers.

Another type of weighted compact nonlinear scheme was constructed in [19]. The scheme was a weighted combination of compact substencils which were two biased third order compact stencils and a central fourth order compact stencil. The scheme would result in a sixth order central compact scheme if with optimal weights for smooth solutions. Ghosh and Baeder employed the idea and developed a new class of compact reconstruction finite difference WENO schemes [13]. In their approach, lower order biased compact stencils were used and the higher order interpolation with optimal weights was upwind. A tridiagonal system was solved at each time step. However, the scheme was high order accurate, essentially non-oscillatory around discontinuities and superior spectral accurate.

Most of the above mentioned compact schemes are based on a finite difference framework. Compact schemes based on a finite volume framework would be more nature especially for unstructured meshes. In [11], Gaitonde et. al. proposed compact-difference-based finite-volume schemes for linear wave phenomena. Kobayashi extended the work [11] to a class of Padé finite volume methods [20]. A fourth-order finite volume compact scheme was provided for the incompressible Navier-Stokes equations in [30] and applied to incompressible

Navier-Stokes equations on staggered grids [17] and compressible Navier-Stokes equations on nonuniform grids [12]. Piller and Stalio developed compact finite volume schemes for one and two dimensional transport and Navier-Stokes equations on stagger grids [31] and for three dimensional scalar advection-diffusion equation on boundary-fitted grids [32], among many others.

In this paper, we follow [13] to develop a finite volume compact WENO (FVCW) scheme, where lower order compact stencils based on cell averages are used. The scheme combines the nonlinear WENO weights to yield a fifth order upwind compact finite volume interpolation. As a new ingredient, we incorporate the MPP polynomial rescaling limiter [41, 43], which would be essentially important for some extreme problems with complex structures. A similar idea with positivity preserving (PP) limiter [43] for one dimensional compressible Euler system has been explored in [15]. For the FVCW scheme with the MPP limiter, numerical experiments will be presented for one and two dimensional scalar hyperbolic equations with application to incompressible Euler equations. The numerical results will show the good performance, maximum principle preserving and high resolution property of our new proposed approach.

The rest of the paper is organized as follows. In Section 2, the finite volume compact WENO scheme is presented for the scalar conservation law in one dimension and the maximum principle satisfying limiter is introduced. The scheme for two dimensional conservation law on rectangular meshes will be described in Section 3 and the application to incompressible flows will be discussed in Section 4. In Section 5, we will show the numerical results for scalar conservation problems and incompressible Euler equations. Finally the conclusions are made in Section 6.

2 Maximum-principle-satisfying finite volume compact WENO schemes

2.1 Finite volume compact WENO scheme

We first briefly review the finite volume compact WENO scheme [15] for one dimensional hyperbolic conservation equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad (2.1)$$

with the initial condition $u(0, x) = u_0(x)$. The computational domain $[a, b]$ is divided into N cells

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b.$$

The cell is denoted by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. For simplicity, we consider an uniform grid and the size of the cell is $\Delta x = \frac{b-a}{N}$. The cell-averaged value of cell j , which denotes to be \bar{u}_j , can be defined as

$$\bar{u}(x_j, t) = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t) dx. \quad (2.2)$$

We approximate (2.1) by the following finite volume conservative scheme

$$\frac{d\bar{u}_j(t)}{dt} = -\frac{1}{\Delta x} (\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}), \quad (2.3)$$

where the numerical flux $\hat{f}_{j+\frac{1}{2}}$ is defined by

$$\hat{f}_{j+\frac{1}{2}} = h(u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+). \quad (2.4)$$

Let $u_{j+\frac{1}{2}}^-$ denote a fifth order approximation of the nodal value $u(x_{j+\frac{1}{2}}, t^n)$ in cell I_j and $u_{j+\frac{1}{2}}^+$ denote a fifth order approximation of the nodal value $u(x_{j+\frac{1}{2}}, t^n)$ from cell I_{j+1} . In this paper, $u_{j+\frac{1}{2}}^-$ and $u_{j+\frac{1}{2}}^+$ are obtained from a high order compact WENO reconstruction, which will be discussed in the following.

An optimal fifth-order compact upwind scheme [33] can be written as

$$\frac{3}{10}u_{j-\frac{1}{2}}^- + \frac{6}{10}u_{j+\frac{1}{2}}^- + \frac{1}{10}u_{j+\frac{3}{2}}^- = \frac{1}{30}\bar{u}_{j-1} + \frac{19}{30}\bar{u}_j + \frac{10}{30}\bar{u}_{j+1}. \quad (2.5)$$

The classical small length scale finite volume linear compact scheme (2.5) is very accurate and keep good resolutions in smooth regions. However, nonphysical oscillations are generated when they are directly applied to problems with discontinuities and the amplitude does not decrease as refining the grid. (2.5) is a weighted combination of three third order compact substencils, they are

$$\frac{2}{3}u_{j-\frac{1}{2}}^- + \frac{1}{3}u_{j+\frac{1}{2}}^- = \frac{1}{6}(\bar{u}_{j-1} + 5\bar{u}_j), \quad (2.6a)$$

$$\frac{1}{3}u_{j-\frac{1}{2}}^- + \frac{2}{3}u_{j+\frac{1}{2}}^- = \frac{1}{6}(5\bar{u}_j + \bar{u}_{j+1}), \quad (2.6b)$$

$$\frac{2}{3}u_{j+\frac{1}{2}}^- + \frac{1}{3}u_{j+\frac{3}{2}}^- = \frac{1}{6}(\bar{u}_j + 5\bar{u}_{j+1}). \quad (2.6c)$$

and to get (2.5) the optimal linear weights are $c_0 = \frac{2}{10}, c_1 = \frac{5}{10}, c_2 = \frac{3}{10}$.

If we replace the optimal linear weights $\{c_0, c_1, c_2\}$ with nonlinear weights $\{\omega_0, \omega_1, \omega_2\}$, a fifth-order finite volume compact WENO scheme can be obtained [15]

$$\begin{aligned} & \frac{2\omega_0 + \omega_1}{3} u_{j-\frac{1}{2}}^- + \frac{\omega_0 + 2(\omega_1 + \omega_2)}{3} u_{j+\frac{1}{2}}^- + \frac{1}{3} \omega_2 u_{j+\frac{3}{2}}^- \\ &= \frac{1}{6} \omega_0 \bar{u}_{j-1} + \frac{5(\omega_0 + \omega_1) + \omega_2}{6} \bar{u}_j + \frac{\omega_1 + 5\omega_2}{6} \bar{u}_{j+1}. \end{aligned} \quad (2.7)$$

A set of nonlinear weights ω_k can be taken as [4]

$$\omega_k = \frac{\alpha_k^z}{\sum_{l=0}^2 \alpha_l^z}, \quad \alpha_k^z = c_k \left(1 + \left(\frac{\tau_5}{\beta_k^z + \epsilon} \right)^p \right), \quad k = 0, 1, 2, \quad (2.8)$$

where $p \geq 1$ is the power parameter. ϵ is a small positive number to avoid the denominator to be 0. In our numerical tests, we take $p = 2$ and $\epsilon = 10^{-13}$.

The smooth indicators β_k^z are chosen from the WENO-Z scheme [3]

$$\beta_k^z = \left(\frac{\beta_k + \epsilon}{\beta_k + \tau_5 + \epsilon} \right), \quad k = 0, 1, 2, \quad (2.9)$$

where $\tau_5 = |\beta_2 - \beta_0|$, which can improve the order of accuracy around the smooth extrema as compared to the classic WENO scheme [18]. The classical smooth indicators β_k ($k = 0, 1, 2$) in [18] are given by

$$\begin{aligned} \beta_0 &= \frac{13}{12} (\bar{u}_{j-2} - 2\bar{u}_{j-1} + \bar{u}_j)^2 + \frac{1}{4} (\bar{u}_{j-2} - 4\bar{u}_{j-1} + 3\bar{u}_j)^2, \\ \beta_1 &= \frac{13}{12} (\bar{u}_{j-1} - 2\bar{u}_j + \bar{u}_{j+1})^2 + \frac{1}{4} (\bar{u}_{j-1} - \bar{u}_{j+1})^2, \\ \beta_2 &= \frac{13}{12} (\bar{u}_j - 2\bar{u}_{j+1} + \bar{u}_{j+2})^2 + \frac{1}{4} (3\bar{u}_j - 4\bar{u}_{j+1} + \bar{u}_{j+2})^2. \end{aligned}$$

We need to solve a tridiagonal system of (2.7) to get $u_{j+\frac{1}{2}}^-$ at each time step since the nonlinear weights depend on the solutions. $u_{j+\frac{1}{2}}^+$ can be obtained similarly. See [15] for more discussions.

For the high-order compact scheme (2.7), we need to set appropriate boundary closures due to the global nature of the reconstruction where all the flux values are involved at each time step. Here for the scheme near boundaries, a fifth-order WENO approximation is used [13].

2.2 Maximum-principle-satisfying limiter

In [15] a fifth-order finite volume compact WENO scheme with positivity-preserving limiter was proposed for solving compressible Euler equations. For scalar hyperbolic conservation

laws (1.1), a similar idea with the polynomial rescaling limiter [41, 43] for preserving the maximum principle is incorporated into the finite volume compact WENO scheme. The Euler forward temporal discretization for the semi-discrete scheme (2.3) is

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda[h(u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) - h(u_{j-\frac{1}{2}}^-, u_{j-\frac{1}{2}}^+)], \quad (2.10)$$

where $\lambda = \Delta t / \Delta x$. $u_{j+\frac{1}{2}}^-$ and $u_{j+\frac{1}{2}}^+$ are the high order approximations of $u(x_{j+\frac{1}{2}}, t^n)$ from the left and right limits which are reconstructed from the finite volume compact WENO scheme. For simplicity, let $m = \min_x u(x, 0)$ and $M = \max_x u(x, 0)$, the polynomial rescaling limiter proposed in [43] can be written as

$$\tilde{p}_j(x) = \theta(p_j(x) - \bar{u}_j) + \bar{u}_j, \quad \theta = \min \left\{ \left| \frac{M - \bar{u}_j^n}{M_j - \bar{u}_j^n} \right|, \left| \frac{m - \bar{u}_j^n}{m_j - \bar{u}_j^n} \right|, 1 \right\}, \quad (2.11)$$

with

$$M_j = \max\{p_j(x_j^*), u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+\}, \quad m_j = \min\{p_j(x_j^*), u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+\}. \quad (2.12)$$

$p_j(x)$ can be seen as a reconstructed polynomial with degree 4 from the cell-averaged values $\{\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}\}$ and two boundary values $\{u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-\}$ in cell I_j for a fifth order approximation. There exists a point x_j^* in cell I_j such that

$$p_j(x_j^*) = \frac{\bar{u}_j^n - \hat{\omega}_1 u_{j-\frac{1}{2}}^+ - \hat{\omega}_G u_{j+\frac{1}{2}}^-}{1 - 2\hat{\omega}_1}. \quad (2.13)$$

$\hat{\omega}_1$ and $\hat{\omega}_G$ are the first and last weights of an G -point Gauss-Lobatto quadrature rule. Let $\tilde{u}_{j-\frac{1}{2}}^+ = \tilde{p}_j(x_{j-\frac{1}{2}})$ and $\tilde{u}_{j+\frac{1}{2}}^- = \tilde{p}_j(x_{j+\frac{1}{2}})$, we get a revised scheme of (2.10)

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda[h(\tilde{u}_{j+\frac{1}{2}}^-, \tilde{u}_{j+\frac{1}{2}}^+) - h(\tilde{u}_{j-\frac{1}{2}}^-, \tilde{u}_{j-\frac{1}{2}}^+)]. \quad (2.14)$$

The scheme (2.14) satisfies a strict maximum principle for scalar conservation laws under the CFL condition

$$\lambda\alpha \leq \hat{\omega}_1. \quad (2.15)$$

with a global Lax-Friedrichs flux $h(u, v) = \frac{1}{2}[f(u) + f(v) - \alpha(v - u)]$, where $\alpha = \max_u |f'(u)|$. In the present compact scheme, although $u_{j+\frac{1}{2}}^-$ and $u_{j-\frac{1}{2}}^+$ are obtained globally which are different from those in [41], the constructed polynomial $p_j(x)$ can be seen locally. Thus this limiter does not destroy the high order accuracy, see the proof in [43, 41] for more details and see [15] for more discussions.

2.3 Temporal discretization

In the present work, a strong stability preserving (SSP) high order Runge-Kutta time discretizations [14] can be used to improve the temporal accuracy in (2.14). A third-order SSP Runge-Kutta method is given as

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n), \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}), \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}), \end{aligned} \quad (2.16)$$

where $L(u)$ is the spatial operator. For a multi-stage SSP Runge-Kutta time method, the MPP limiter will be applied and the tridiagonal system (2.7) will be solved at each stage of each time step.

3 Two dimensional case

In this section, we consider the finite volume compact WENO scheme for solving the two dimensional conservation law

$$u_t + f(u)_x + g(u)_y = 0, \quad (3.1)$$

on the domain $[a, b] \times [c, d]$ with rectangular meshes

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_x + \frac{1}{2}} = b, \quad c = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N_y + \frac{1}{2}} = d. \quad (3.2)$$

Denoting $\Delta x = (b - a)/N_x$ and $\Delta y = (d - c)/N_y$ for uniform sizes, the finite volume scheme for (3.1) on cell $I_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ can be obtained as follows

$$\begin{aligned} \bar{u}_{ij}^{n+1} &= \bar{u}_{ij}^n - \frac{\Delta t}{\Delta x \Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} [\hat{f}(u_{i+\frac{1}{2},j}^-(y), u_{i+\frac{1}{2},j}^+(y)) - \hat{f}(u_{i-\frac{1}{2},j}^-(y), u_{i-\frac{1}{2},j}^+(y))] dy \\ &\quad - \frac{\Delta t}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [\hat{g}(u_{i,j+\frac{1}{2}}^-(x), u_{i,j+\frac{1}{2}}^+(x)) - \hat{g}(u_{i,j-\frac{1}{2}}^-(x), u_{i,j-\frac{1}{2}}^+(x))] dx, \end{aligned} \quad (3.3)$$

where $u_{i+\frac{1}{2},j}^-(y)$, $u_{i-\frac{1}{2},j}^+(y)$, $u_{i,j+\frac{1}{2}}^-(x)$ and $u_{i,j-\frac{1}{2}}^+(x)$ denote the traces of fifth-order polynomial on the four edges of cell $I_{i,j}$ respectively. The cell-averaged value of cell $I_{i,j}$, which denotes to be \bar{u}_{ij}^n , can be defined as

$$\bar{u}_{ij}(t) = \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x, y, t) dx dy. \quad (3.4)$$

We will use the Lax-Friedrichs flux for \hat{f} and \hat{g}

$$\hat{f}(u, v) = \frac{1}{2}[f(u) + f(v) - a_1(v - u)], \quad a_1 = \max_u |f'(u)|, \quad (3.5a)$$

$$\hat{g}(u, v) = \frac{1}{2}[g(u) + g(v) - a_2(v - u)], \quad a_2 = \max_u |g'(u)|. \quad (3.5b)$$

The integrals in (3.3) are approximated by the Gaussian quadrature rule as in [41] with sufficient accuracy. By using an L -point Gaussian quadrature rule, we can get an approximation for (3.3)

$$\begin{aligned} \bar{u}_{ij}^{n+1} = & \bar{u}_{ij}^n - \lambda_1 \sum_{\beta=1}^L \omega_\beta [\hat{f}(u_{i+\frac{1}{2},\beta}^-, u_{i+\frac{1}{2},\beta}^+) - \hat{f}(u_{i-\frac{1}{2},\beta}^-, u_{i-\frac{1}{2},\beta}^+)] \\ & - \lambda_2 \sum_{\beta=1}^L \omega_\beta [\hat{g}(u_{\beta,j+\frac{1}{2}}^-, u_{\beta,j+\frac{1}{2}}^+) - \hat{g}(u_{\beta,j-\frac{1}{2}}^-, u_{\beta,j-\frac{1}{2}}^+)], \end{aligned} \quad (3.6)$$

where $\lambda_1 = \frac{\Delta t}{\Delta x}$, $\lambda_2 = \frac{\Delta t}{\Delta y}$, $u_{i\pm\frac{1}{2},\beta}^\mp = u_{i\pm\frac{1}{2},j}^\mp(y_j^\beta)$, $u_{\beta,j\pm\frac{1}{2}}^\mp = u_{i,j\pm\frac{1}{2}}^\mp(x_i^\beta)$, y_j^β denotes the Gaussian quadrature points on $[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ and x_i^β denotes the Gaussian quadrature points on $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, ω_β is the corresponding quadrature weight.

The algorithm for maximum-principle-satisfying fifth-order finite volume compact WENO scheme solving (3.1) consists of the following steps:

1. The fifth-order finite volume compact WENO scheme (2.7) is used to get the four edge averages $\{\bar{u}_{i+\frac{1}{2},j}^-, \bar{u}_{i+\frac{1}{2},j}^+\}$ for fixed j and $\{\bar{u}_{i,j+\frac{1}{2}}^-, \bar{u}_{i,j+\frac{1}{2}}^+\}$ for fixed i , e.g., $\bar{u}_{i+\frac{1}{2},j}^- = \frac{1}{\Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x_{i+\frac{1}{2}}^-, y) dy$, similarly for others. Values at the quadrature points $\{u_{i+\frac{1}{2},\beta}^-, u_{i+\frac{1}{2},\beta}^+\}$ and $\{u_{\beta,j+\frac{1}{2}}^-, u_{\beta,j+\frac{1}{2}}^+\}$ are obtained by the fifth-order WENO schemes [18].
2. Let $m = \min_{x,y} u(x, y, 0)$ and $M = \max_{x,y} u(x, y, 0)$, the maximum-principle-satisfying limiter is constructed as follows [43]

$$\theta_{ij} = \min \left\{ \left| \frac{M - \bar{u}_{ij}^n}{M_{ij} - \bar{u}_{ij}^n} \right|, \left| \frac{m - \bar{u}_{ij}^n}{m_{ij} - \bar{u}_{ij}^n} \right|, 1 \right\}, \quad (3.7)$$

where

$$M_{ij} = \max\{p_{ij}(x_i^*, y_j^*), u_{i\mp\frac{1}{2},\beta}^\pm, u_{\beta,j\mp\frac{1}{2}}^\pm\}, \quad m_{ij} = \min\{p_{ij}(x_i^*, y_j^*), u_{i\mp\frac{1}{2},\beta}^\pm, u_{\beta,j\mp\frac{1}{2}}^\pm\}, \quad (3.8)$$

and there exists a point (x_i^*, y_j^*) in cell I_{ij} such that

$$p_{ij}(x_i^*, y_j^*) = \frac{\bar{u}_{ij}^n - \sum_{\beta=1}^L \omega_\beta \hat{\omega}_1 [\mu_1(u_{i+\frac{1}{2},\beta}^- + u_{i-\frac{1}{2},\beta}^+) + \mu_2(u_{\beta,j+\frac{1}{2}}^- + u_{\beta,j-\frac{1}{2}}^+)]}{1 - 2\hat{\omega}_1}. \quad (3.9)$$

here $\mu_1 = \lambda_1 a_1 / (\lambda_1 a_1 + \lambda_2 a_2)$ and $\mu_2 = \lambda_2 a_2 / (\lambda_1 a_1 + \lambda_2 a_2)$ with a_1, a_2 defined in (3.5). Note that ω_β is the weight of an L -point Gaussian quadrature rule in (3.6) and $\hat{\omega}_1$ is the first weight of an G -point Gauss-Lobatto quadrature with $G = 4$ for a fifth order scheme.

3. Finally $u_{i \mp \frac{1}{2}, \beta}^\pm$ and $u_{\beta, j \mp \frac{1}{2}}^\pm$ in (3.6) are updated by

$$u_{i \mp \frac{1}{2}, \beta}^\pm = \theta_{ij}(u_{i \mp \frac{1}{2}, \beta}^{\pm, old} - \bar{u}_{ij}^n) + \bar{u}_{ij}^n, \quad u_{\beta, j \mp \frac{1}{2}}^\pm = \theta_{ij}(u_{\beta, j \mp \frac{1}{2}}^{\pm, old} - \bar{u}_{ij}^n) + \bar{u}_{ij}^n. \quad (3.10)$$

where $u_{i \mp \frac{1}{2}, \beta}^{\pm, old}$ and $u_{\beta, j \mp \frac{1}{2}}^{\pm, old}$ are the values in (3.6) before modified.

The CFL condition for the two dimensional case with the MPP limiter is taken to be

$$a_1 \lambda_1 + a_2 \lambda_2 \leq \hat{\omega}_1. \quad (3.11)$$

Similarly as the one dimensional case, the limiter does not destroy the high order accuracy. The proof can be referred to [41].

4 Application to two dimensional incompressible flows

We now consider the incompressible Euler equations in the vorticity stream-function formulation [2],

$$\omega_t + (u\omega)_x + (v\omega)_y = 0, \quad (4.1a)$$

$$\Delta\psi = \omega, (u, v) = (-\phi_y, \phi_x), \quad (4.1b)$$

with $\omega(x, y, 0) = \omega_0(x, y)$, $(u, v) \cdot \mathbf{n} = \text{given on } \partial\Omega$.

The divergence-free condition $u_x + v_y = 0$ can be obtained from equation (4.1b), which implies (4.1a) is equivalent to the non-conservative form

$$\omega_t + u\omega_x + v\omega_y = 0. \quad (4.2)$$

The conservative equation (4.1a) itself does not imply the maximum principle $\omega(x, y, t) \in [m, M]$ if without the incompressibility condition $u_x + v_y = 0$, where $m = \min_{x,y} \omega(x, y, 0)$ and $M = \max_{x,y} \omega(x, y, 0)$. This is the main difficulty to get a maximum-principle-satisfying scheme for solving (4.1) [41]. It has been proved in [41] that a high-order DG scheme [26] for (4.1) with the MPP limiter under a suitable CFL condition satisfies the maximum principle without destroying the high order accuracy. In the following, we will consider the finite

volume compact WENO scheme on rectangular meshes similarly as in Section 3 for solving (4.1a).

The finite volume scheme with Euler forward time discretization for (4.1a) on cell $I_{i,j}$ is

$$\begin{aligned}\bar{u}_{ij}^{n+1} = & \bar{u}_{ij}^n - \frac{\Delta t}{\Delta x \Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} [\hat{h}(\omega_{i+\frac{1}{2},j}^-(y), \omega_{i+\frac{1}{2},j}^+(y), u_{i+\frac{1}{2},j}(y)) - \hat{h}(\omega_{i-\frac{1}{2},j}^-(y), \omega_{i-\frac{1}{2},j}^+(y), u_{i-\frac{1}{2},j}(y))] dy \\ & - \frac{\Delta t}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [\hat{h}(\omega_{i,j+\frac{1}{2}}^-(x), \omega_{i,j+\frac{1}{2}}^+(x), v_{i,j+\frac{1}{2}}(x)) - \hat{h}(\omega_{i,j-\frac{1}{2}}^-(x), \omega_{i,j-\frac{1}{2}}^+(x), v_{i,j-\frac{1}{2}}(x))] dx,\end{aligned}\quad (4.3)$$

where $\omega_{i+\frac{1}{2},j}^-(y)$, $\omega_{i-\frac{1}{2},j}^+(y)$, $\omega_{i,j+\frac{1}{2}}^-(x)$ and $\omega_{i,j-\frac{1}{2}}^+(x)$ denote the traces of a fifth-order polynomial on the four edges of cell $I_{i,j}$ respectively. The cell-averaged value of the vorticity on cell $I_{i,j}$, which denotes to be $\bar{\omega}_{ij}^n$, can be defined as

$$\bar{\omega}_{ij}(t) = \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \omega(x, y, t) dx dy. \quad (4.4)$$

The Lax-Friedrichs upwind biased flux is used in this work, for instance on the right edge

$$\hat{h}(\omega_{i+\frac{1}{2},j}^-(y), \omega_{i+\frac{1}{2},j}^+(y), u_{i+\frac{1}{2},j}(y)) = \frac{1}{2} [u_{i+\frac{1}{2},j}(y)(\omega_{i+\frac{1}{2},j}^-(y) + \omega_{i+\frac{1}{2},j}^+(y)) - \alpha(\omega_{i+\frac{1}{2},j}^+(y) - \omega_{i+\frac{1}{2},j}^-(y))], \quad (4.5)$$

where α is the maximum of $|u_{i+\frac{1}{2},j}(y)|$ either locally or globally.

By using the L -point Gaussian quadrature rule, an approximation for (4.3) can be written as

$$\begin{aligned}\bar{u}_{ij}^{n+1} = & \bar{u}_{ij}^n - \lambda_1 \sum_{\beta=1}^L \omega_{\beta} [\hat{h}(\omega_{i+\frac{1}{2},\beta}^-, \omega_{i+\frac{1}{2},\beta}^+, u_{i+\frac{1}{2},\beta}) - \hat{h}(\omega_{i-\frac{1}{2},\beta}^-, \omega_{i-\frac{1}{2},\beta}^+, u_{i-\frac{1}{2},\beta})] \\ & - \lambda_2 \sum_{\beta=1}^L \omega_{\beta} [\hat{h}(\omega_{\beta,j+\frac{1}{2}}^-, \omega_{\beta,j+\frac{1}{2}}^+, v_{\beta,j+\frac{1}{2}}) - \hat{h}(\omega_{\beta,j-\frac{1}{2}}^-, \omega_{\beta,j-\frac{1}{2}}^+, v_{\beta,j-\frac{1}{2}})].\end{aligned}\quad (4.6)$$

The algorithm for maximum-principle-satisfying fifth-order finite volume compact WENO scheme in Section 3 can be applied to (4.6). Under the CFL condition $a_1 \lambda_1 + a_2 \lambda_2 \leq \frac{1}{2} \min_{\alpha=1,\dots,G} \hat{\omega}_{\alpha}$, the scheme (4.6) satisfies the maximum principle and maintains the high order accuracy [41]. For incompressible Euler equations in the vorticity stream-function formulation (4.1), the Poisson equation (4.1b) is solved by the Fourier spectral method.

5 Numerical examples

In this section, we provide some classical numerical examples for the fifth-order finite volume compact WENO scheme with the third order SSP Runge-Kutta time discretization (2.16),

which is denoted as the “FVCW” scheme. The original fifth order WENO scheme of Jiang and Shu [18] is denoted as “WENO-JS” and the fifth-order WENO-Z scheme [4] will be denoted as “WENO-Z”. We will compare the present FVCW scheme with WENO-JS and WENO-Z schemes in some of the following test cases. We compute the solutions up to time T on a mesh of N and $N \times N$ with the CFL conditions (2.15) and (3.11) for one and two dimensional cases respectively. The minimum and maximum numerical cell-averaged values are denoted as $(\bar{u}_h)_{min}$ and $(\bar{u}_h)_{max}$ or $(\bar{\omega}_h)_{min}$ and $(\bar{\omega}_h)_{max}$ for the incompressible flow problems respectively.

5.1 One-dimensional test cases

Example 5.1. We first solve the linear advection equation

$$u_t + u_x = 0, \quad u(x, 0) = u_0(x), \quad (5.1)$$

with periodic boundary conditions on the domain $[0, 2]$. We take the smooth initial data $u_0(x) = 0.5 + \sin^4(2\pi x)$ to test the accuracy and the maximum principle preserving property. In order to compare our numerical results with those obtained by the finite volume WENO schemes in [43], the weights in [18] are used. The L^1 and L^∞ errors and orders at time $T = 0.1$ with and without limiters are given in Table 5.1. We observe that the numerical solutions obtained with limiters are all lied in $[0.5, 1.5]$, while the minimum values might be less than 0.5 if without limiters. It shows the present FVCW scheme with limiters satisfies the strict discrete maximum principle and the high order of accuracy is maintained. The numerical results are comparable to those obtained by the finite volume WENO scheme in Table 5.1 in [41].

Example 5.2. We then consider the linear advection equation (5.1) with the following initial condition [18]

$$u(x, 0) = \begin{cases} \frac{1}{6}(G(x, z - \delta) + G(x, z + \delta) + 4G(x, z)), & x \in (-0.8, -0.6), \\ 1, & x \in (-0.4, -0.2), \\ 1 - |10(x - 0.1)|, & x \in (0, 0.2), \\ \frac{1}{6}(F(x, a - \delta) + F(x, a + \delta) + 4F(x, a)), & x \in (0.4, 0.6), \\ 0, & \text{otherwise,} \end{cases} \quad (5.2)$$

where $G(x, z) = e^{-\beta(x-z)^2}$, $F(x, \gamma) = \sqrt{\max(1 - \alpha^2(x - \gamma)^2, 0)}$, $z = -0.7$, $\delta = 0.0005$, $\alpha = 10$, $\beta = \frac{\log 2}{36\delta^2}$, $\gamma = 0.5$. The computational domain is $[-1, 1]$ with periodic boundary conditions.

Table 5.1: L^1 and L^∞ errors and orders for Example 5.1 with $u_0(x) = 0.5 + \sin^4(2\pi x)$.

N	L^1 error	Order	L^∞ error	Order	$(\bar{u}_h)_{min}$	$(\bar{u}_h)_{max}$
with limiters						
20	5.56E-03		1.23E-02		0.5056844053	1.4336458051
40	9.02E-04	2.62	2.95E-03	2.06	0.5005823859	1.4838817402
80	5.28E-05	4.09	2.68E-04	3.46	0.5000324661	1.4959125035
160	8.81E-07	5.91	6.96E-06	5.27	0.5000001415	1.4989730706
320	2.10E-08	5.39	1.19E-07	5.87	0.5000000287	1.4997430328
640	4.82E-10	5.44	1.75E-09	6.10	0.5000000018	1.4999357475
without limiters						
20	4.80E-03		1.38E-02		0.4988356090	1.4364151979
40	1.01E-03	2.24	4.04E-03	1.77	0.4996159401	1.4844567924
80	6.73E-05	3.91	3.35E-04	3.59	0.4999199982	1.4959203562
160	8.83E-07	6.25	6.89E-06	5.60	0.4999999714	1.4989731984
320	2.11E-08	5.39	1.20E-07	5.85	0.5000000279	1.4997430383
640	4.82E-10	5.45	1.75E-09	6.10	0.5000000018	1.4999357479

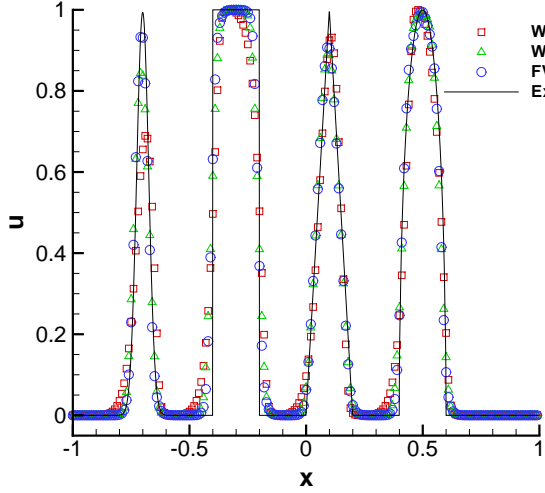
Table 5.2: Maximum and minimum numerical solutions for Example 5.2 at $T = 8$.

N	with limiters		without limiters	
	$(\bar{u}_h)_{min}$	$(\bar{u}_h)_{max}$	$(\bar{u}_h)_{min}$	$(\bar{u}_h)_{max}$
50	6.2532042145E-05	0.9844087972	-6.4754491922E-02	1.0136701866
100	6.2444287122E-08	0.9985999188	-4.1188418939E-03	1.0332676399
200	5.9311237190E-13	0.9999770077	-9.5377767652E-05	1.0043551468
400	6.6787958057E-19	0.9999998937	-1.2991894432E-07	1.0000004916
800	6.0843658046E-31	1.0000000000	-9.7496437457E-08	1.0000000859

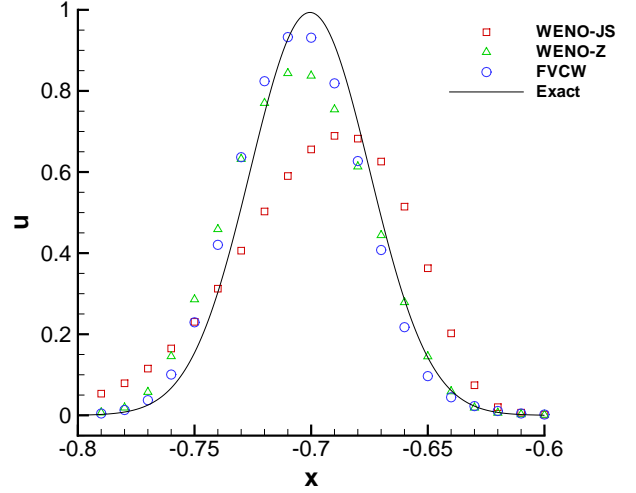
The solution is a travelling wave formed by the combination of a Gaussian, a square wave, a sharp triangle wave and a half ellipse. After a period of $T = 2$, the solution will get back to its initial position. In Fig. 5.1, we show the solution at time $T = 8$ on a grid with $N = 200$ for the WENO-JS, WENO-Z and FVCW schemes. In the zoom-in figures, we can see the FVCW scheme captures better results than the other two schemes.

The maximum and minimum numerical solutions are listed in Table 5.2. The numerical solutions with limiters are all within the range $[0, 1]$. However, without limiters the minimum values are negative and the maximum values are great than 1.

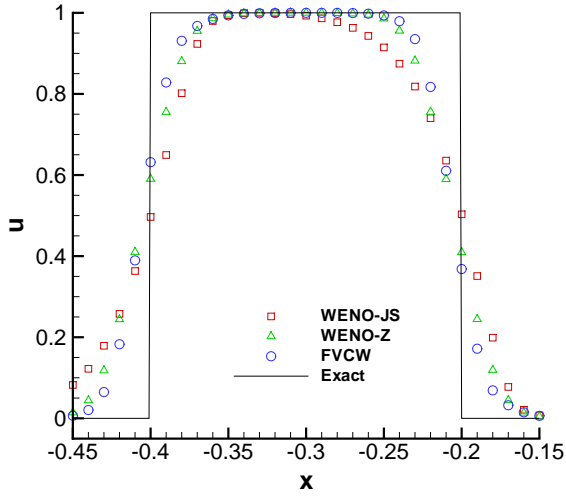
Example 5.3. This example is the linear advection equation (5.1) with initial condition



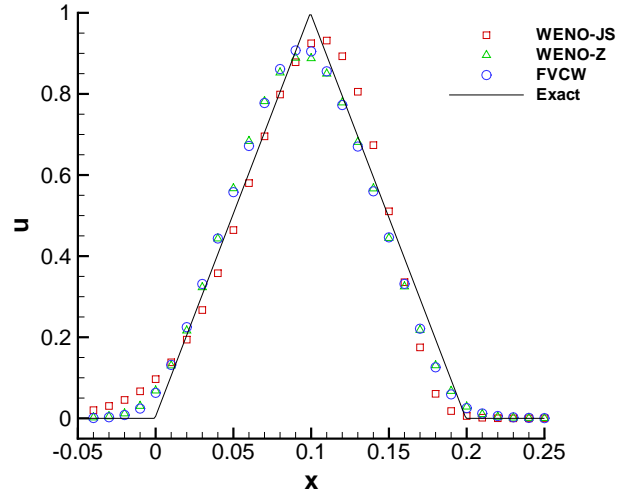
(a) $T=8$



(b) Gaussian wave



(c) Square wave



(d) Triangular wave

Figure 5.1: Numerical results computed by the WENO-JS, WENO-Z and FVCW schemes with the exact solution for Example 5.2 at $T = 8$. $N = 200$.

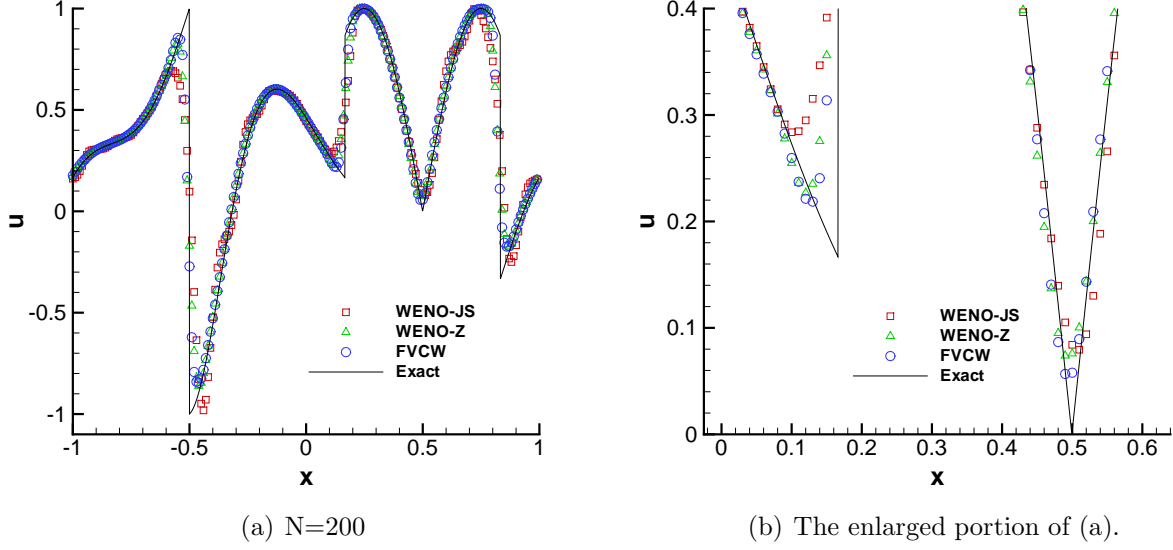


Figure 5.2: Numerical results computed by the WENO-JS, WENO-Z and FVCW schemes with the exact solution for Example 5.3 at $T = 2$. $N = 200$.

[36],

$$u(x + 0.5, 0) = \begin{cases} -x \sin(\frac{3}{2}\pi x^2), & -1 \leq x < \frac{1}{3}, \\ |\sin(2\pi x)|, & |x| \leq \frac{1}{3}, \\ 2x - 1 - \frac{1}{6}\sin(3\pi x), & \frac{1}{3} < x \leq 1, \end{cases} \quad (5.3)$$

on the domain $[-1, 1]$ with periodic boundary conditions. This test case is used to check how the FVCW scheme can capture smooth and discontinuous solution structures. In Fig. 5.2, we show the solution at time $T = 2$ on the grid with $N = 200$ for the WENO-JS, WENO-Z and FVCW schemes. From the figure we can see the numerical solutions all capture the discontinuities without oscillations and match the accurate solution in the smooth regions very well. In the zoom-in Fig. 5.2(b), it shows that the FVCW scheme performs better than the other two schemes.

Example 5.4. We now solve the nonlinear Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad u(x, 0) = u_0(x), \quad (5.4)$$

with periodic boundary conditions. We test our scheme with the initial condition $u_0(x) = \sin^4(x)$ on $[0, 2\pi]$. At $T = 0.5$, the solution is smooth. The L^1 and L^∞ errors and orders for the FVCW scheme with and without limiters are given in Tables 5.3. For this nonlinear problem, the numerical solutions with limiters are all within the range $[-0.5, 1.5]$ and the 5th order of accuracy is also maintained. At $T = 1.2$ the solution develops a shock. The

Table 5.3: L^1 and L^∞ errors and orders for Example 5.4 with $u_0(x) = \sin^4(x)$.

N	L^1 error	Order	L^∞ error	Order	$(\bar{u}_h)_{min}$	$(\bar{u}_h)_{max}$
with limiters						
20	7.11E-03		1.75E-02		4.8027078317E-03	0.9661788333
40	1.54E-03	2.21	9.32E-03	0.91	5.9499022915E-04	0.9860591424
80	1.79E-04	3.11	2.08E-03	2.17	5.3042057661E-05	0.9984210631
160	9.62E-06	4.21	1.88E-04	3.47	4.2567524841E-06	0.9995883857
320	3.54E-07	4.76	7.63E-06	4.62	3.2493209900E-07	0.9999342274
640	1.10E-08	5.00	2.36E-07	5.02	1.6373928322E-08	0.9999490507
without limiters						
20	6.34E-03		1.85E-02		1.7218752333E-04	0.9664450834
40	1.48E-03	2.10	9.06E-03	1.03	-3.1070308046E-05	0.9860690299
80	1.83E-04	3.02	2.08E-03	2.12	-1.0538041762E-05	0.9984240105
160	9.57E-06	4.25	1.88E-04	3.47	-1.4739089684E-06	0.9995889053
320	3.51E-07	4.77	7.63E-06	4.62	-1.6598308446E-07	0.9999343533
640	1.14E-08	4.95	2.36E-07	5.02	-1.6619301474E-08	0.9999490606

numerical solution and the exact solution are shown in Fig. 5.3. With limiters, the minimum numerical value is 5.25E-006. However, it is -2.84E-006 if without limiters.

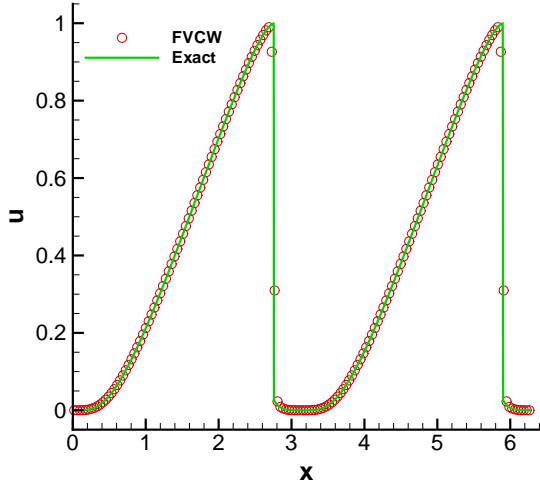
Example 5.5. The nonlinear Buckley-Leverett problem [36] is used for reservoir simulation,

$$u_t + f(u)_x = 0, \quad f(u) = \frac{4u^2}{4u^2 + (1-u)^2}. \quad (5.5)$$

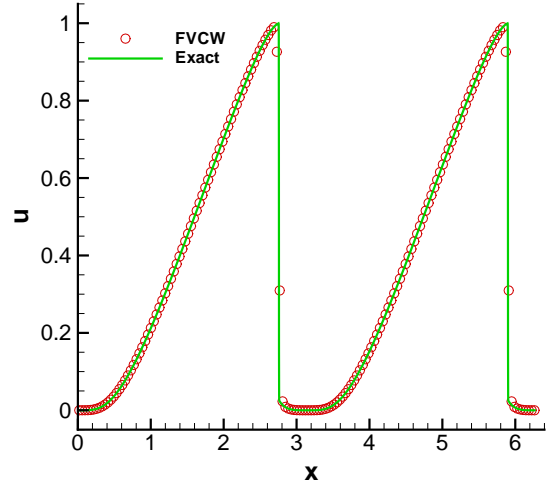
$f(u)$ is a nonconvex function and the initial condition is

$$u(x, 0) = \begin{cases} 1, & -\frac{1}{2} < x < 0, \\ 0, & \text{otherwise.} \end{cases} \quad (5.6)$$

We compute the solution up to $T = 0.4$ on the domain $[-1, 1]$ with inflow and outflow boundary conditions on each side respectively. The numerical solutions of WENO-JS, WENO-Z, FVCW with $N = 100$ and the exact solution are shown in Fig. 5.4. The zoom-in Fig. 5.4(b) shows that a slightly more accurate result can be obtained by the FVCW scheme. The maximum and minimum numerical solutions are listed in Table 5.4 for the FVCW scheme with and without limiters. The numerical solutions with limiters are all within the range $[0, 1]$ as expected.

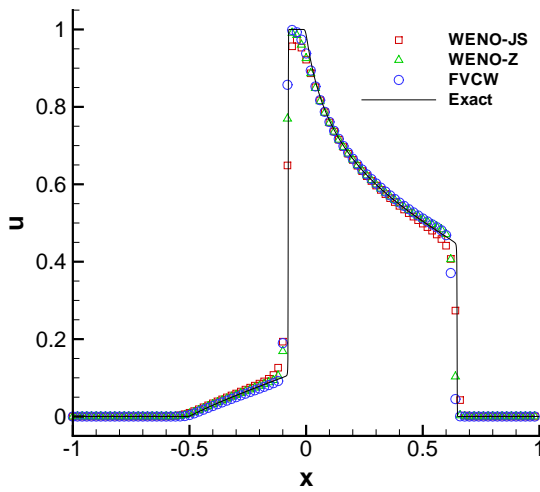


(a) without limiters

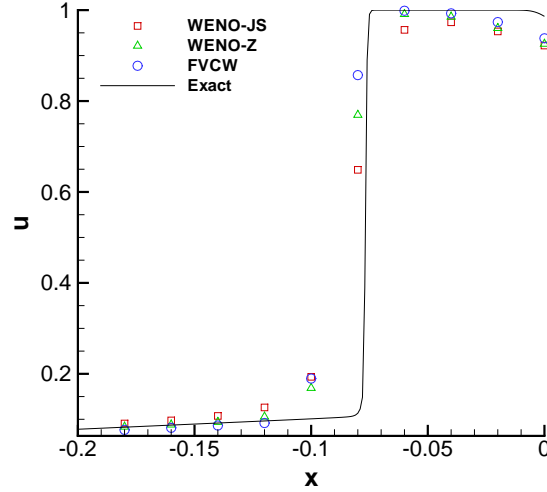


(b) with limiters

Figure 5.3: Numerical results with the exact solution for Example 5.4 at $T = 1.2$ with $N = 160$.



(a) $N=100$



(b) The enlarged portion of (a).

Figure 5.4: Numerical results computed by the WENO-JS, WENO-Z and FVCW with the exact solution for Example 5.5 at $T = 0.4$ with $N = 100$.

Table 5.4: Maximum and minimum numerical solutions for Example 5.5 at $T = 0.4$.

	with limiters		without limiters	
N	$(\bar{u}_h)_{min}$	$(\bar{u}_h)_{max}$	$(\bar{u}_h)_{min}$	$(\bar{u}_h)_{max}$
50	2.3624923015E-13	0.9000000000	-1.0004055700E-08	0.9000000000
100	5.3108644691E-21	0.9876234859	-1.2483926241E-08	0.9876241852
200	1.9526892745E-30	0.9991089796	-1.5844261578E-08	0.9991089584
400	1.3188489112E-47	0.9999947633	-1.8249582807E-08	0.9999947651
800	7.5182650962E-74	1.0000000000	-1.7783953062E-08	1.0000000166

Table 5.5: L^1 and L^∞ errors and orders for Example 5.6 with $u_0(x, y) = \sin^4(2\pi(x + y))$.

N \times N	L^1 error	Order	L^∞ error	Order	$(\bar{u}_h)_{min}$	$(\bar{u}_h)_{max}$
with limiters						
10 \times 10	5.81E-02		1.09E-01		3.2175761831E-02	0.9000000000
20 \times 20	4.24E-03	3.78	1.33E-02	3.03	3.0806089368E-03	0.9611971380
40 \times 40	5.78E-04	2.87	2.25E-03	2.57	3.2800792678E-04	0.9916153520
80 \times 80	2.90E-05	4.32	2.21E-04	3.35	2.4330912515E-05	0.9979365718
160 \times 160	5.32E-07	5.77	3.78E-06	5.87	2.1151199593E-07	0.9994852045
320 \times 320	1.00E-08	5.73	1.01E-07	5.23	1.1731843775E-08	0.9998714151
without limiters						
10 \times 10	5.17E-02		8.22E-02		-4.6206453729E-02	0.9000000000
20 \times 20	4.10E-03	3.66	1.50E-02	2.46	-1.0767640535E-02	0.9652658048
40 \times 40	4.78E-04	3.10	1.72E-03	3.12	-6.4056253913E-04	0.9918385835
80 \times 80	2.84E-05	4.07	2.29E-04	2.91	-7.1932689350E-05	0.9979464519
160 \times 160	5.05E-07	5.81	3.78E-06	5.92	-3.2339904232E-08	0.9994861246
320 \times 320	8.04E-09	5.97	3.27E-08	6.85	5.9884217874E-09	0.9998715025

5.2 Two-dimensional test cases

Example 5.6. We first solve the two dimensional linear advection equation

$$u_t + u_x + u_y = 0, \quad u(x, y, 0) = u_0(x, y), \quad (x, y) \in [0, 1] \times [0, 1], \quad (5.7)$$

with periodic boundary condition and a smooth initial data $u_0(x, y) = \sin^4(2\pi(x + y))$. We compute the solution up to time $t = 0.1$. The L^1 and L^∞ errors and orders for the FVCW scheme with and without limiters are given in Table 5.5. The FVCW scheme with limiters satisfies the strict maximum principle and the 5th order of accuracy is maintained, which is similar to the one dimensional case.

Example 5.7. We solve the nonlinear Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = 0, \quad u(x, y, 0) = u_0(x, y), \quad (x, y) \in [0, 2\pi] \times [0, 2\pi], \quad (5.8)$$

Table 5.6: L^1 and L^∞ errors and orders for Example 5.7 with $u_0(x, y) = \sin^4(x + y)$

$N \times N$	L^1 error	Order	L^∞ error	Order	$(\bar{u}_h)_{min}$	$(\bar{u}_h)_{max}$
without limiters						
20×20	1.70E-03		1.22E-02		-2.7162407531E-003	0.9729124291
40×40	1.89E-04	3.17	2.43E-03	2.33	-6.7740627680E-005	0.9813145773
80×80	1.97E-05	3.26	5.51E-04	2.14	-9.6911213559E-006	0.9979503187
160×160	8.80E-07	4.48	5.57E-05	3.31	-7.5156683063E-007	0.9993992055
320×320	4.24E-08	4.37	3.07E-06	4.18	-8.4287588702E-008	0.9997704325
640×640	1.35E-09	4.97	1.25E-07	4.62	-1.7786892420E-008	0.9999547580
with limiters						
20×20	2.37E-03		1.14E-02		1.4611105650E-003	0.9676804528
40×40	2.96E-04	3.00	1.96E-03	2.54	2.3919879074E-004	0.9819309191
80×80	2.96E-05	3.32	5.60E-04	1.81	1.4266765832E-005	0.9978612888
160×160	1.27E-06	4.56	5.57E-05	3.33	1.7175042462E-007	0.9993913736
320×320	6.26E-08	4.33	3.07E-06	4.18	1.5078276150E-008	0.9997697244
640×640	2.44E-09	4.68	1.26E-07	4.61	2.4266981729E-009	0.9999546678

with periodic boundary condition.

We also test our scheme with the initial condition $u_0(x, y) = \sin^4(x + y)$ on $[0, 2\pi] \times [0, 2\pi]$ with periodic boundary conditions. We compute up to $T = 0.2$ when the solution is still smooth. The L^1 and L^∞ errors and orders of accuracy for the FVCW scheme with and without limiters are given in Table 5.6, almost fifth order accuracy can be observed for this example. At time $T = 0.8$ when the solution develops a still shock, the numerical solutions with 160×160 grid points and the exact solution are showed in Fig. 5.5. With limiters, the minimum numerical value is 1.02E-006 while it is -2.52E-002 if without limiters.

Example 5.8. The two-dimensional inviscid Buckley-Leverett equation with gravitational effects in the y direction [21] can be written as

$$u_t + f(u)_x + g(u)_y = 0, \quad (x, y) \in [-1.5, 1.5] \times [-1.5, 1.5], \quad (5.9)$$

where

$$f(u) = \frac{u^2}{u^2 + (1 - u)^2}, \quad g(u) = f(u)(1 - 5(1 - u)^2). \quad (5.10)$$

and the initial condition is

$$u(x, y, 0) = \begin{cases} 1, & x^2 + y^2 < 0.5, \\ 0, & \text{otherwise.} \end{cases} \quad (5.11)$$

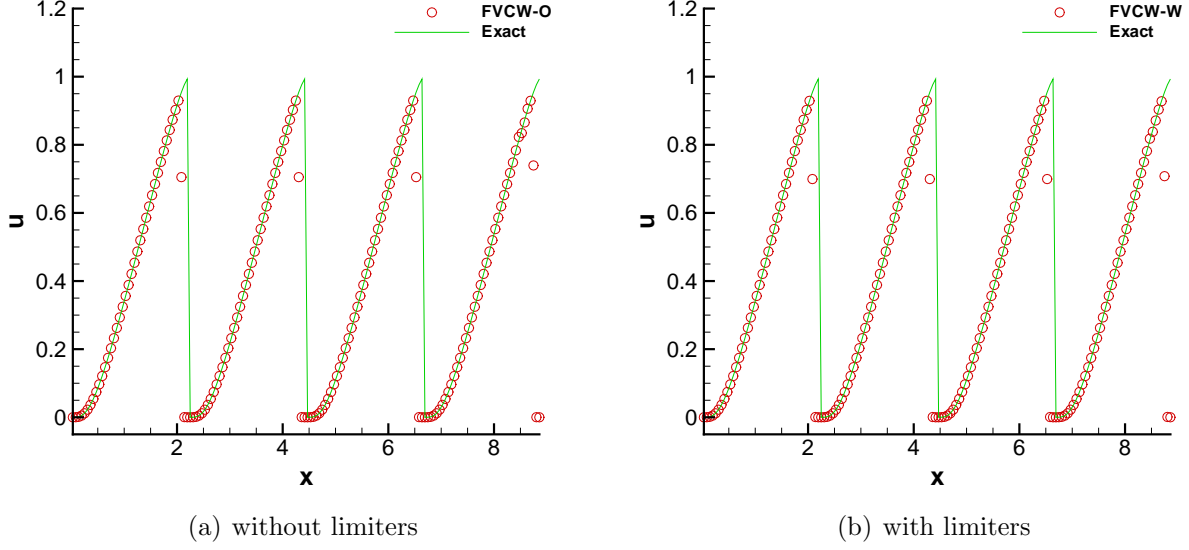


Figure 5.5: Numerical results for Example 5.7 with $u_0(x, y) = \sin^4(x + y)$ at $T = 0.8$. Mesh 160×160 . Cuts along $x = y$.

Table 5.7: Maximum and minimum numerical solutions for Example 5.8 at $T = 0.5$.

$N \times N$	with limiters		without limiters	
	$(\bar{u}_h)_{min}$	$(\bar{u}_h)_{max}$	$(\bar{u}_h)_{min}$	$(\bar{u}_h)_{max}$
8×8	5.2763420443E-004	0.9182487023	-5.3290980853E-002	1.0218414115
16×16	4.9239390613E-007	0.9514705351	-2.5761986049E-002	0.9554234787
32×32	7.8904099515E-012	0.9840518251	-4.7749433037E-003	0.9841847426
64×64	8.6575061287E-017	0.9971065091	-2.8304959899E-007	0.9971278020
128×128	1.3048650350E-025	0.9998762415	-1.6326732186E-008	0.9998768328
128×128	5.3604284574E-041	0.9999998498	-1.8832845160E-008	0.9999998751

We compute the solution up to time $T = 0.5$ with periodic boundary conditions. The maximum and minimum numerical solutions are listed in Table 5.7 for the 5th order FVCW scheme with and without limiters. Without limiters, we can observe obvious undershoots. They are completely eliminated by the scheme with limiters. The numerical solution of FVCW scheme at $T = 0.5$ with $N \times N = 128 \times 128$ are shown in Fig. 5.6, which is similar to the result in [21]. The contour plot in Fig. 5.6 shows that the present scheme produces high resolution numerical solutions without significant spurious oscillations.

5.3 Incompressible flow

In this section, both the FVCW scheme and the finite volume compact (FVC) scheme (which is the FVCW scheme with optimal linear weights) are considered. The FVCW scheme with

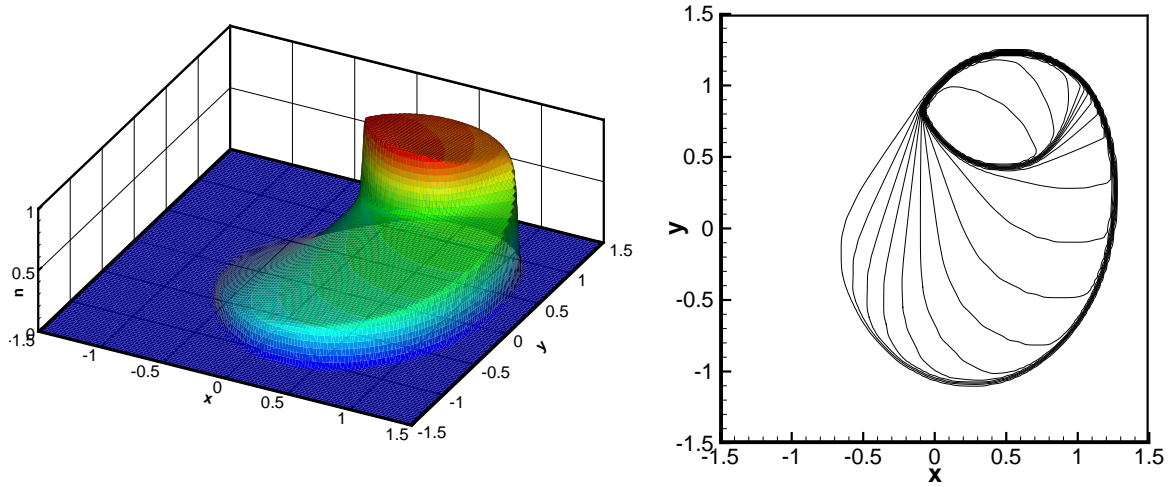


Figure 5.6: Numerical solution for Example 5.8 with FVCW scheme at $T=0.5$ on a 128×128 mesh.

limiters is denoted as “FVCW-W” and it is denoted as “FVCW-O” if without limiters. Similarly, the FVC scheme with and without limiters are denoted as “FVC-W” and “FVC-O”.

Example 5.9. (Rigid body rotation) Consider the following rigid body rotation problem [24]

$$\omega_t + (-(y - 0.5)\omega)_x + ((x - 0.5)\omega)_y = 0, \quad (x, y) \in [0, 1] \times [0, 1]. \quad (5.12)$$

The initial profile consists of a smooth hump, a cone and a slotted cylinder [24, 29] as in Fig. 5.7. The numerical solution of the FVCW scheme with limiters after one period of evolution on a mesh of $N \times N = 100 \times 100$ is shown in Fig. 5.8. The numerical solutions of the FVCW scheme and the FVC scheme with and without limiters at $T = 2\pi$ are plotted in Fig. 5.9, by comparing with the exact solutions. With limiters, the minimum numerical value of the FVCW scheme is 4.46E-015 and the maximum value is 0.99984. However, they are -6.24E-002 and 1.05022 if without limiters. The numerical results are similar to those in [34].

Example 5.10. (Swirling deformation flow) The swirling deformation flow [24] is

$$\omega_t + (\sin^2(\pi x) \sin(2\pi y)g(t)\omega)_x + (-\sin^2(\pi y) \sin(2\pi x)g(t)\omega)_y = 0, \quad (x, y) \in [0, 1] \times [0, 1],$$

with $g(t) = \cos(\pi t/T)$ on the time interval $0 \leq t \leq T$. The initial condition is the same as in Fig. 5.7.

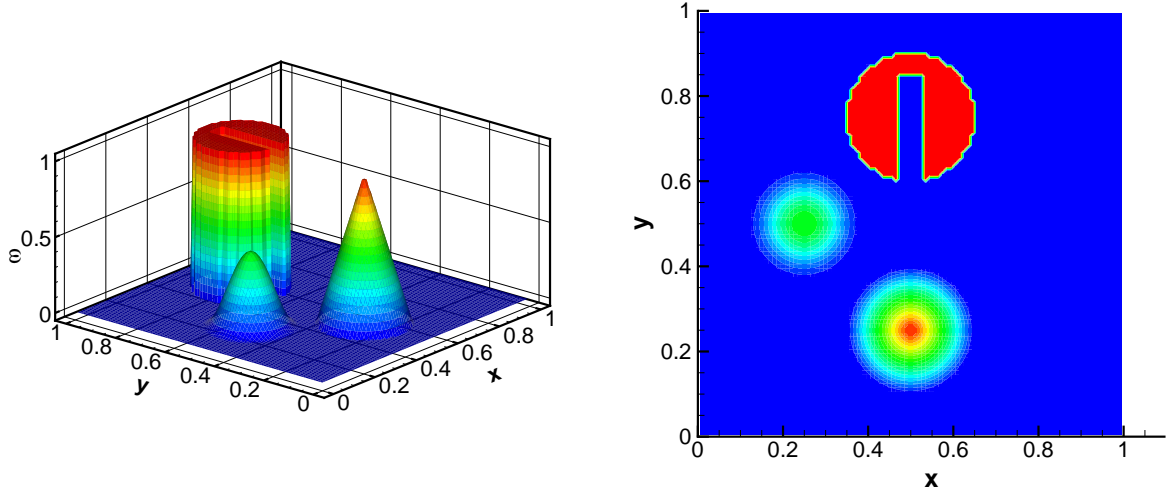


Figure 5.7: Initial profile.

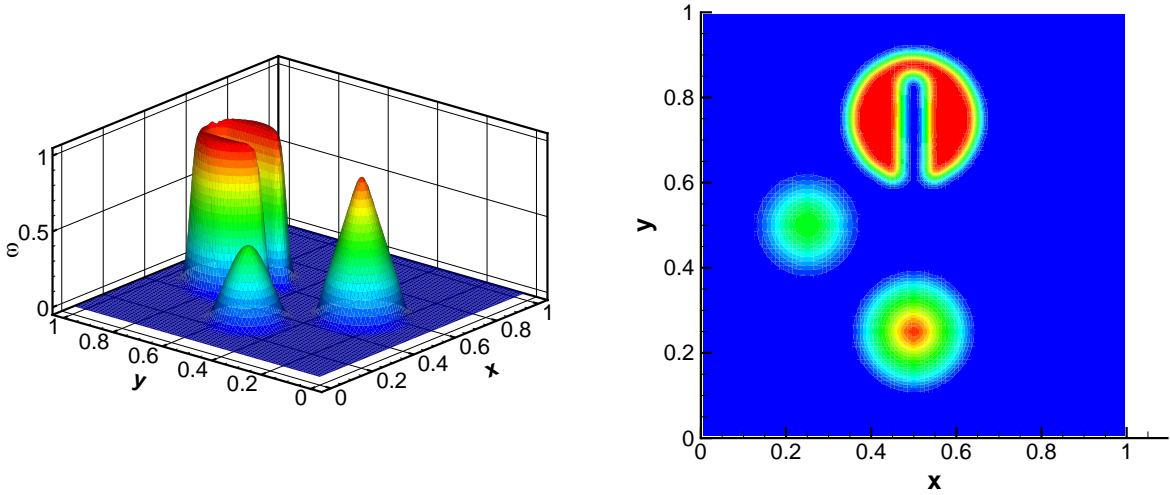
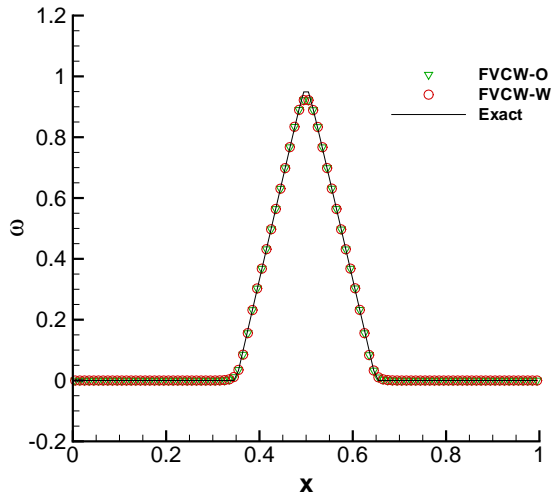
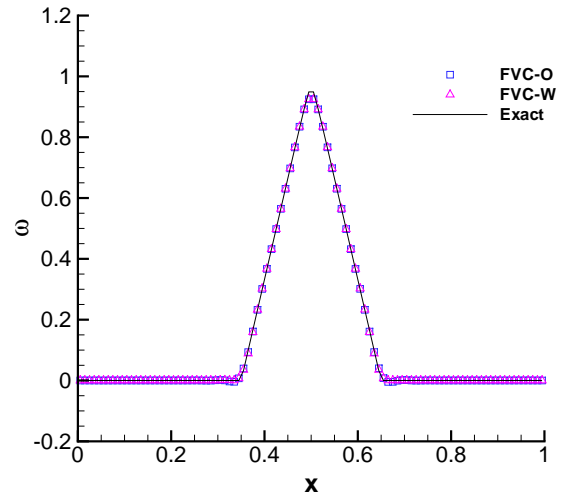


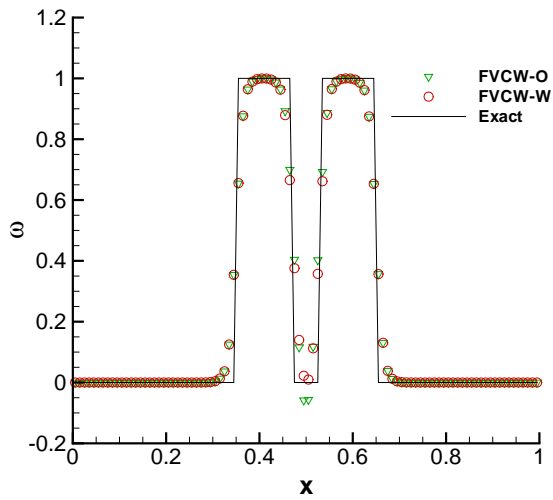
Figure 5.8: Numerical results for Example 5.9 at $T = 2\pi$ with limiters.



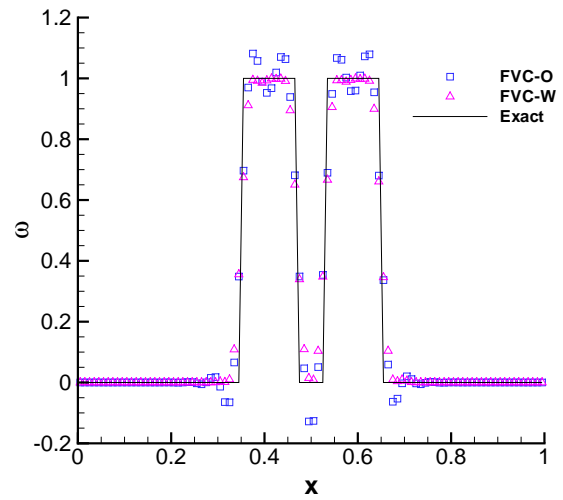
(a) $y = 0.25$



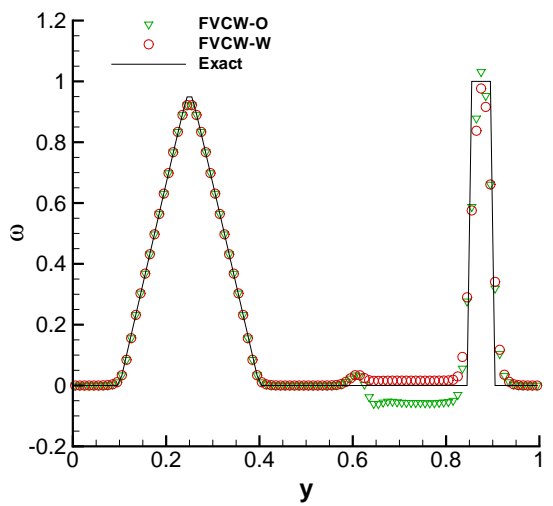
(b) $y = 0.25$



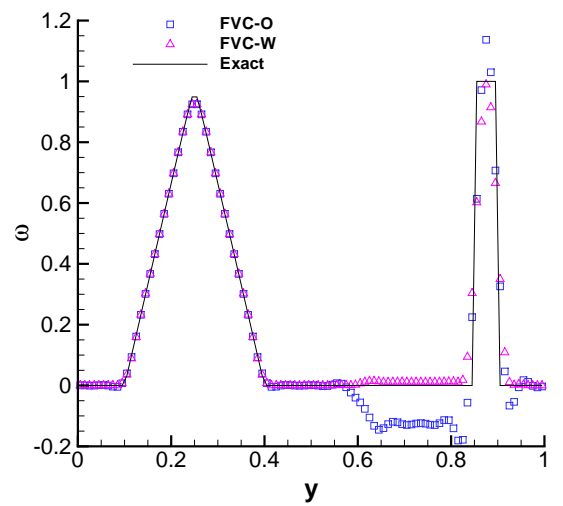
(c) $y = 0.75$



(d) $y = 0.75$



(e) $x = 0.5$



(f) $x = 0.5$

Figure 5.9: Cross-sections of the numerical solutions with the exact solutions for Example 5.9 at $T = 0.2$. Left: FVCW; Right: FVC-O.

The flow slows down and reverses direction when the initial data is recovered at time T . We take $T = 1.5$. The numerical solution of the FVCW scheme on a mesh of $N \times N = 100 \times 100$ at $T = 1.5$ is shown in Fig. 5.10. Similarly the cuts of the numerical solution at T are plotted in Fig. 5.11 by comparing with the exact solutions. With limiters, the minimum numerical value is 3.50E-018 and the maximum value is 0.99967. However, they are -6.76E-002 and 1.09007 if without limiters. At time $T/2$, the solution is quite deformed from the initial data, see Fig. 5.12. The minimum and maximum values with limiters are 5.91E-019 and 0.99903. Without limiters, they are -6.60E-002 and 1.06939 respectively.

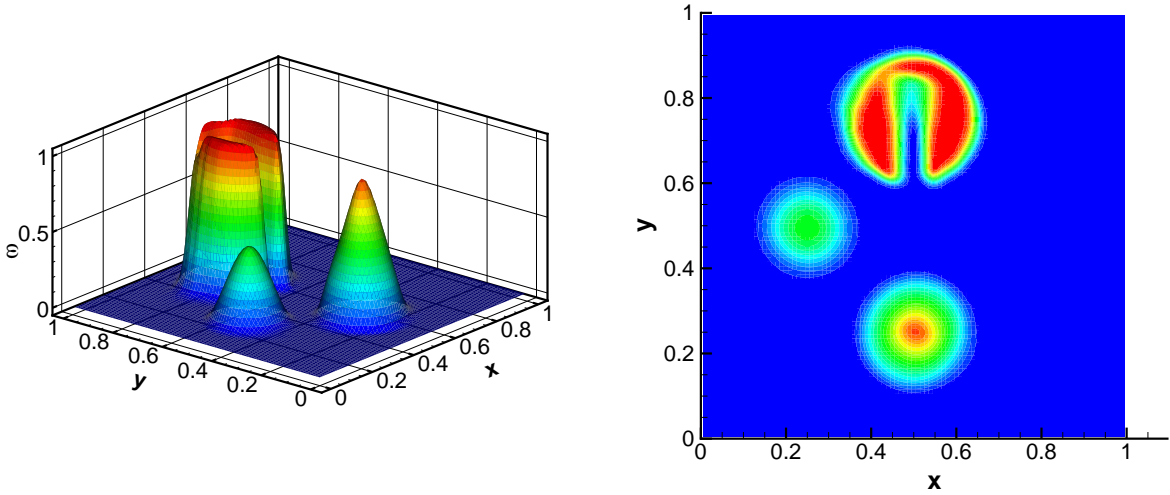


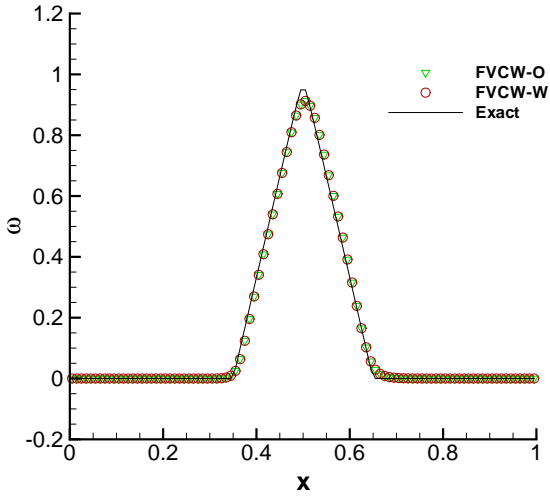
Figure 5.10: Numerical results for Example 5.10 at $T = 1.5$.

Example 5.11. In this example, we consider two-dimensional incompressible equations (4.1). The computational domain is $[0, 2\pi] \times [0, 2\pi]$ with periodic boundary conditions. For this problem with the smooth exact solution $\omega(x, y, t) = -2\sin(x)\sin(y)$, at $T = 1$ the designed 5th order of accuracy can be clearly observed in Table 5.8, and the numerical solutions with limiters are all within the range $[-2, 2]$.

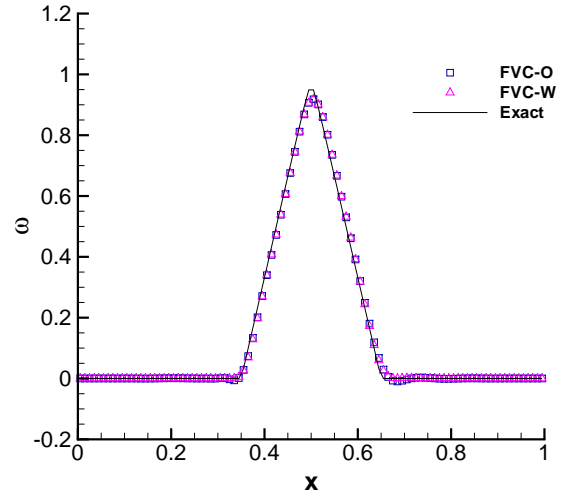
Example 5.12. (The double shear layer problem) We then consider (4.1) for the double shear layer problem on the domain $[0, 2\pi] \times [0, 2\pi]$ with the initial conditions given by

$$\omega(x, y, 0) = \begin{cases} \delta \cos(x) - \frac{1}{\rho} \text{sech}^2\left(\frac{1}{\rho}(y - \frac{\pi}{2})\right), & y \leq \pi, \\ \delta \cos(x) + \frac{1}{\rho} \text{sech}^2\left(\frac{1}{\rho}(\frac{3\pi}{2} - y)\right), & \text{otherwise,} \end{cases} \quad (5.13)$$

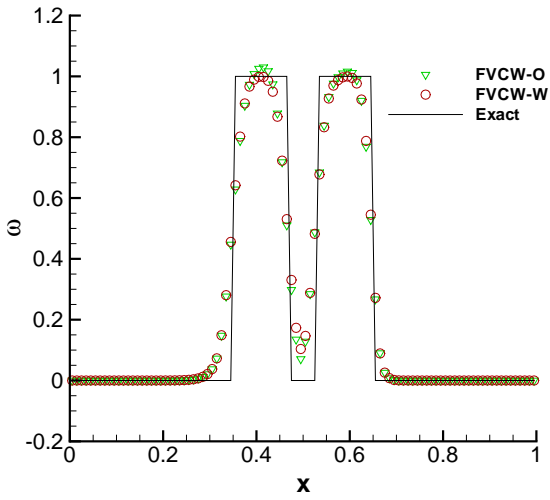
where $\rho = \frac{\pi}{15}$ and $\delta = 0.05$.



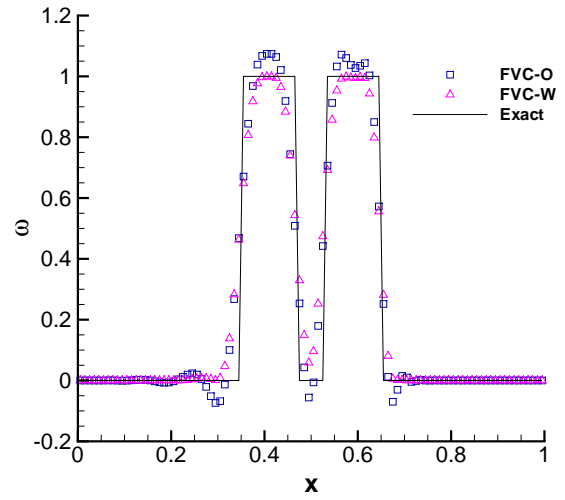
(a) $y = 0.25$



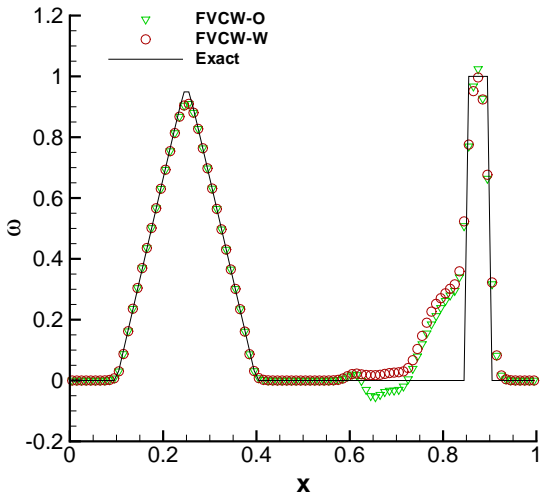
(b) $y = 0.25$



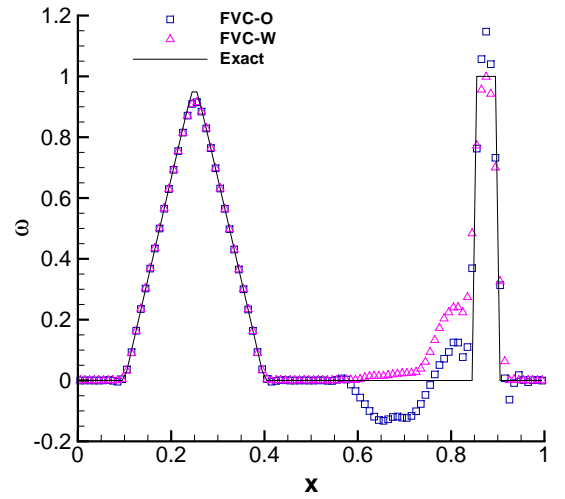
(c) $y = 0.75$



(d) $y = 0.75$



(e) $x = 0.5$



(f) $x = 0.5$

Figure 5.11: Cross-sections of the numerical solutions with the exact solutions for Example 5.10 at $T = 1.5$. Left: FVCW, Right: FVC-O.

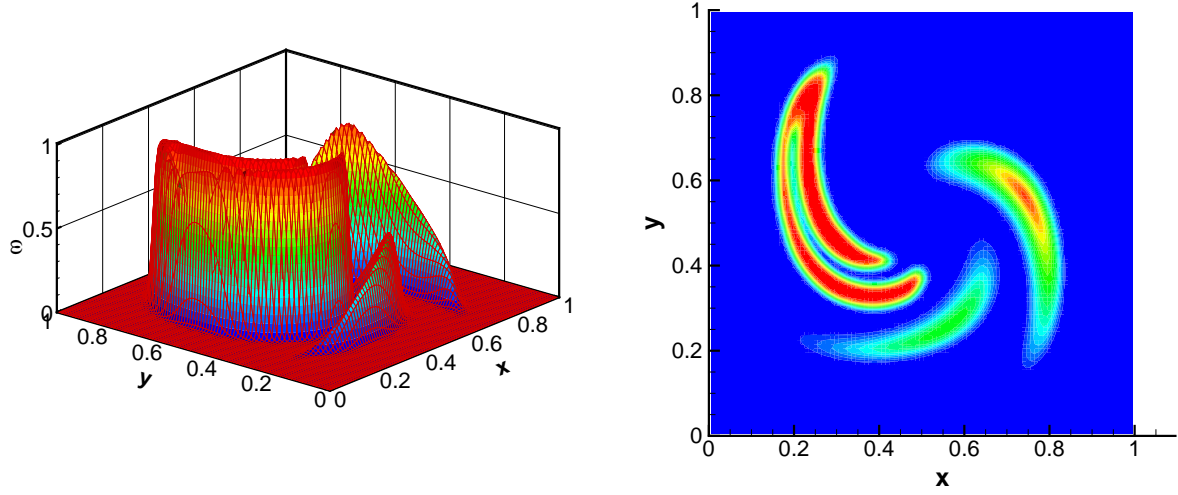


Figure 5.12: The deformed numerical results for Example 5.10 at $T/2$.

Table 5.8: L^1 and L^∞ errors and orders for Example 5.11 with $\omega_0(x, y, 0) = -2 \sin(x) \sin(y)$, with limiters.

$N \times N$	L^1 error	Order	L^∞ error	Order	$(\bar{\omega}_h)_{min}$	$(\bar{\omega}_h)_{max}$
20×20	1.920E-06		8.349E-06		-1.9350632621	1.9350632621
40×40	3.769E-08	5.67	2.706E-07	4.95	-1.9836046769	1.9836046769
80×80	7.533E-10	5.64	8.674E-09	4.96	-1.9958910455	1.9958910455
160×160	1.669E-11	5.50	2.724E-10	4.99	-1.9989721276	1.9989721276
320×320	4.170E-13	5.32	8.524E-12	5.00	-1.9997429923	1.9997429923

The contours of the vorticity at $T = 6$ with meshes of 64×64 and 128×128 for the FVCW scheme with and without limiters are shown in Fig. 5.13. The contours of the vorticity at $T = 8$ are shown in Fig. 5.14. For this problem, we can not see any visible difference of the results between with and without limiters, but the numerical solution with limiters is within the range $[-\delta - \frac{1}{\rho}, \delta + \frac{1}{\rho}]$. The results are similar to those in [41, 26].

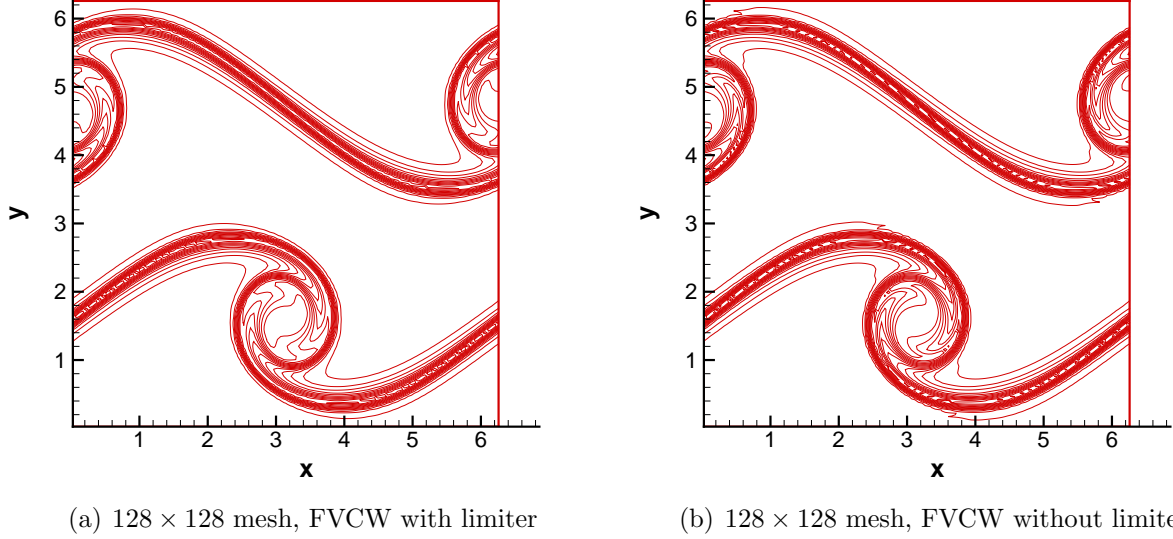
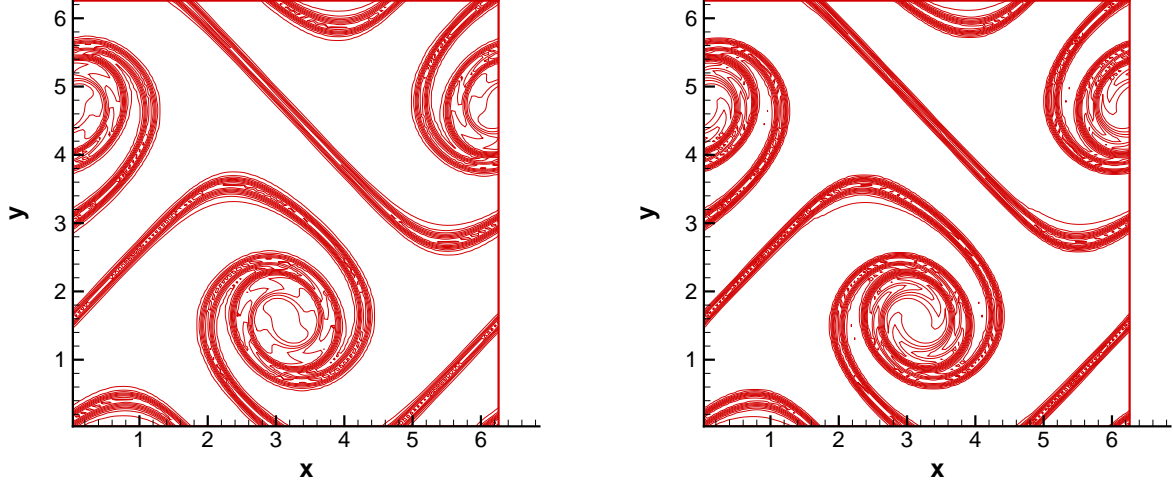


Figure 5.13: Numerical results for Example 5.12 at $T = 6$, 64×64 (left), 128×128 (right).

Example 5.13. (The vortex patch problem) Finally we solve the incompressible Euler equations (4.1) for the vortex patch problem on the domain $[0, 2\pi] \times [0, 2\pi]$ with the initial conditions given by

$$\omega(x, y, 0) = \begin{cases} -1, & \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}, \frac{\pi}{4} \leq y \leq \frac{3\pi}{4}, \\ 1, & \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}, \frac{5\pi}{4} \leq y \leq \frac{7\pi}{4}, \\ 0, & \text{otherwise.} \end{cases} \quad (5.14)$$

In Fig. 5.15, we show the contour plots of vorticity and the cuts along the diagonal at $T = 5$ and $T = 10$ for the FVCW scheme with limiters. The results are also similar to those in [34, 41, 26]. The minimum and maximum numerical values at $T = 5$ with limiters are -0.999995 and 0.999995 , without limiters they are -1.000012 and 1.000012 . We omit the contour plots for the case without limiters due to similarity.



(a) 128×128 mesh, FVCW with limiter

(b) 128×128 mesh, FVCW without limiter

Figure 5.14: Numerical results for Example 5.12 at $T = 8$, 64×64 (left), 128×128 (right).

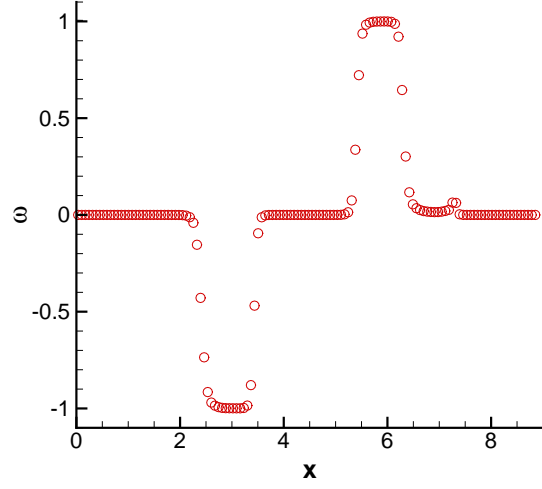
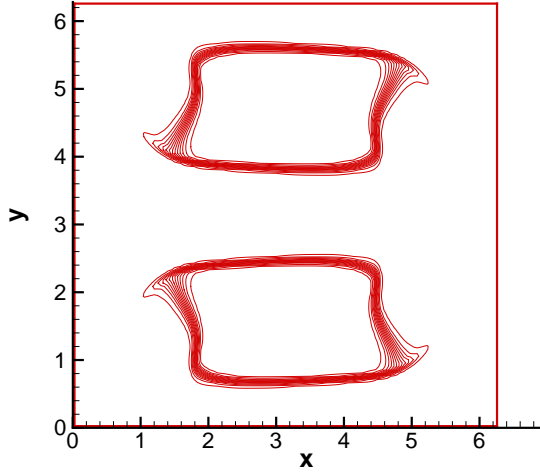
6 Conclusions

In this paper, we developed a maximum-principle-satisfying high order finite volume compact WENO scheme. By applying a polynomial scaling limiter to the finite volume compact WENO scheme at each stage of an explicit Runge-Kutta time method, the scheme satisfies the strict maximum principle under suitable CFL numbers without destroying high order of accuracy. Both one-dimensional and two-dimensional examples including incompressible flow problems are tested, the results showed that the compact scheme has better resolution compared to the classical finite volume non-compact WENO scheme. The application of our proposed method on unstructured meshes is subject to our ongoing work.

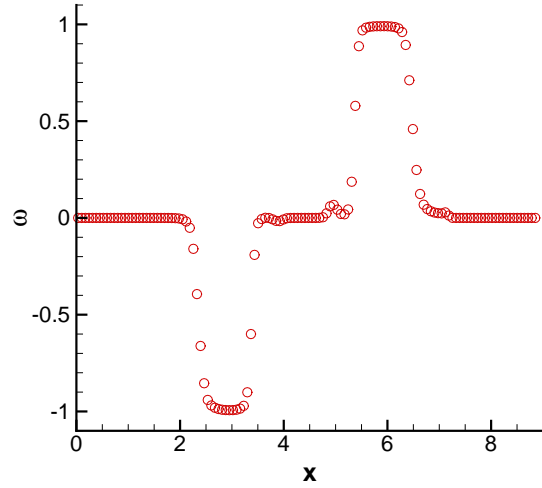
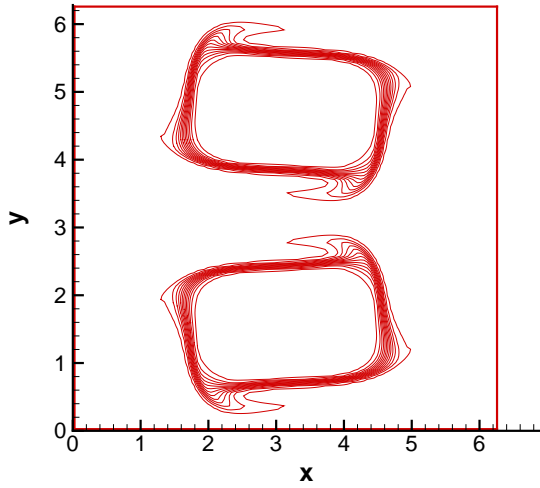
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(a) $T=5$



(b) $T=10$

Figure 5.15: Numerical results of the FVCW scheme for Example 5.13. Mesh 128×128 .

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